Jordan-Wigner Transformations and Their Generalizations for Multidimensional Systems

Martin S. Kochmański

Institute of Physics, University of Rzeszów

T.Rejtana 16 A, 35–310 Rzeszów, Poland

e-mail: mkochma@atena.univ.rzeszow.pl

(February 1, 2008)

Abstract

In the paper nonlinear transformations of the Jordan-Wigner (JW) type are introduced in the form different from the ones known previously, for the purpose of expressing multi-index Pauli operators in terms of multi-index Fermi creation and annihilation operators. These JW transformations in the general case being a subject of a rather complicated algebra of transposition relations between various sets of Fermi creation and annihilation operators, depending on the common multiindex of the latter, is shown. As an example, the two- and three- dimensional transformations of the JW type are investigated, their properties and possible applications in analysis of a couple of lattice models of statistical mechanics and also an example of application of these transformations to problems of self-avoiding walks in graph theory, are discussed. The relation of the obtained transformations to the previously known transformations of the JW type for higher dimensions is shown.

PACS number(s): 05.50+q, 75.10. Jm.

I. INTRODUCTION

In 1928 P. Jordan and E. Wigner have introduced their celebrated transformations [1], which enabled expressing Fermi operators $c_m^\dagger$ and c_m in terms of the Pauli operators for a one-dimensional system. Namely, the following relations are satisfied (we use the notation generally accepted in contemporary literature on the subject):

$$c_m = \exp\left(i\pi \sum_{j=1}^{m-1} \tau_j^+ \tau_j^-\right)\tau_m^-, \quad c_m^\dagger = \exp\left(i\pi \sum_{j=1}^{m-1} \tau_j^+ \tau_j^-\right)\tau_m^+, \tag{1.1}$$

where $\tau_m^\pm$ - Pauli operators [2], satisfying anticommutation transposition relations for one node m of the chain

$$\left\{\tau_m^+, \tau_m^-\right\}_+ = 1, \qquad \left(\tau_m^+\right)^2 = \left(\tau_m^-\right)^2 = 0, \tag{1.2}$$

and commutation transposition relations for different nodes of the chain

$$\left[\tau_m^\pm, \tau_{m'}^\pm\right]_- = 0 \qquad (m \neq m'). \tag{1.3}$$

The Pauli operators $\tau_m^\pm$ can be expressed in terms of the Pauli matrices τ_m^k, $(k = x, y, z)$ [3,4] by the following well known formulae:

$$\tau_m^\pm = \frac{1}{2}\left(\tau_m^z \pm i\tau_m^y\right), \quad \tau_m^x = -2\left(\tau_m^+\tau_m^- - \frac{1}{2}\right), \quad \tau_m^z = \tau_m^+ + \tau_m^-.$$

There exist also transformations that are inverse to (1.1):

$$\tau_m^- = \exp\left(i\pi \sum_{j=1}^{m-1} c_j^\dagger c_j\right)c_m, \quad \tau_m^+ = \exp\left(i\pi \sum_{j=1}^{m-1} c_j^\dagger c_j\right)c_m^\dagger \tag{1.4}$$

and the following relation is satisfed: $\tau_m^+\tau_m^- = c_m^\dagger c_m$.

The Jordan-Wigner (J-W) transformations (1.1), (1.4) are applied widely in numerous domains of quantum as well as classical physics, especially in quantum field theory, statistical mechanics of quantum and classical systems, in physics of phase transitions and critical effects [2,5–13], etc. One of the most spectacular examples of application of Jordan-Wigner transformations is the outstanding paper by Schultz, Mattis and Lieb [12], in which the

authors introduced a new approach to solving the two-dimensional Ising model. The key moment of the paper is application of the J-W transformations (the transfer matrix method was known earlier [5]), and then reduction of the problem to a problem of many fermions on a one-dimensional lattice, i.e. transformation to the fermion representation. The solution [12] of the Lenz-Ising-Onsager problem is, in our opinion, the most beautiful and powerful of all solutions given before and after this one. It seems to be the proper place to mention the note made by J. Ziman [15] p. 209, which sounds a little bit pessimistic. The note concerns the question connected with various approaches to the solution of the Lenz-Ising-Onsager problem after switching on external magnetic field of a finite value. Ziman, reporting the approach introduced in the paper [12], writes that taking into account external magnetic field extremely complicates the operator representation of the transfer matrix in the second quantization representation and that *"Exactly in this point the limitation of the Onsager's method is manifested"*.

We feel that there is some misunderstanding in this statement, because using the Onsager's method we do not apply the field-theoretical language of creation and annihilation operators for fermions (or bosons), but the authors of the paper [12] do. Indeed, the Onsager's method [16], [17] exhibits some limitation concerning its application to solving the two-dimensional Ising model in external magnetic field or in solving the three-dimensional Ising model [3]. In contrast, we deal with a completely different situation considering the approach of Schultz, Mattis and Lieb [12], which uses with all of its beauty the field theoretical language of second quantization. This method should no longer be treated as a beautiful trick but as a powerful tool with great prospects for generalizations. The first step towards such a generalization, leading to the solutions of the Lenz-Ising-Onsager problems for $d = 2$, $H \neq 0$ and for $d = 3$, $H \neq 0$, is (in additionally to the transfer matrix method) the introduction of the J-W transformations, generalized to two-dimensional and three-dimensional systems (see [18,19]).

Further in this paper the transformations (1.1) and (1.4) will be called one-dimensional Jordan-Wigner transformations. It is well known that instead of transformations (1.1), (1.4)

we can introduce transformations of the form:

$$b_m = exp\left(i\pi\sum_{p=m+1}^{\mathfrak{m}}\tau_p^+\tau_p^-\right)\tau_m^- \qquad b_m^\dagger = exp\left(i\pi\sum_{p=m+1}^{\mathfrak{m}}\tau_p^+\tau_p^-\right)\tau_m^+, \tag{1.5}$$

and the transformations inverse to them. These transformations will be called inversional. Operators $(b_m^\dagger, b_m)$ are obviously the Fermi creation and annihilation operators. Operators $(c_m^\dagger, c_m)$ and $(b_m^\dagger, b_m)$ are connected by relations of the type:

$$c_m^\dagger = -(-1)^{\hat{\mathfrak{m}}} b_m^\dagger, \quad c_m = (-1)^{\hat{\mathfrak{m}}} b_m, \quad c_m^\dagger c_m = b_m^\dagger b_m, \quad m = 1, 2, \ldots, \mathfrak{m} \tag{1.6}$$

where: $\hat{\mathfrak{m}} = \sum_m c_m^\dagger c_m = \sum_m b_m^\dagger b_m$ - the operator of the number of all fermions, as it follows from (1.1), (1.4)- (1.5). It is easy to show that

$$[c_m^\dagger, b_{m'}^\dagger]_- = \ldots = [c_m, b_{m'}] = 0, \qquad m \neq m', \qquad\qquad [c_m^\dagger, b_m]_- = -(-1)^{\hat{\mathfrak{m}}}, \tag{1.7}$$

i.e. these operators commute for different m.

Recently appeared a few papers [22–25], investigating generalization of J-W transformations for lattice systems to higher dimensions. Fradkin [22] and Y.R. Wang [23] consider generalization to the two-dimensional case (2D), and Huerta and Zanelli [24] and S. Wang [25] - to the three-dimensional case (3D). The latter two authors show also that appropriate generalizations to the 4D and higher dimensional cases is straightforward. Further in the paper we refer to some results of the papers [24,25] and especially from the paper of S. Wang [25]. Therefore let us remind shortly the most important points of the paper. In the paper [25] were given solutions to the equations (here we preserve original notation of the author):

$$S^-(\mathbf{x}) = U(\mathbf{x})c(\mathbf{x}), \quad S^+(\mathbf{x}) = c^\dagger(\mathbf{x})U^+(\mathbf{x}), \quad U(\mathbf{x}) = \exp\left[i\pi\sum_{\mathbf{z}} w(\mathbf{x},\mathbf{z})c^\dagger(\mathbf{z})c(\mathbf{z})\right] \tag{1.8}$$

where $S^\pm(\mathbf{x})$ - Pauli operators, $c^\dagger(\mathbf{x}), c(\mathbf{x})$ - Fermi operators, and the function $w(\mathbf{x},\mathbf{z})$ should satisfy the following condition:

$$e^{i\pi w(\mathbf{z},\mathbf{x})} = -e^{i\pi w(\mathbf{x},\mathbf{z})}, \qquad \mathbf{x} \neq \mathbf{z} \tag{1.9}$$

The solution of the equation (1.9) is of the form: for the 1D case $w(x,z) = \Theta(x - z)$, where $\Theta(x)$ – the step-like function (the unit - valued Heaviside function), and for the 2D case

$$w(\mathbf{x}, \mathbf{z}) = \Theta(x_1 - z_1)(1 - \delta_{x_1 z_1}) + \Theta(x_2 - z_2)\delta_{x_1 z_1} \tag{1.10}$$

where $x_{1,2}$ and $z_{1,2}$ - components of the vectors $\mathbf{x}$ and $\mathbf{z}$, respectively, in a chosen coordinate system $(\mathbf{e}_1, \mathbf{e}_2)$, and δ_{xz} -the one-dimensional lattice delta function. Finally, for the 3D case S. Wang [25] writes the solution:

$$\begin{aligned} w(\mathbf{x}, \mathbf{z}) &= \Theta(x_1 - z_1)(1 - \delta_{x_1, z_1}) + \Theta(x_2 - z_2)\delta_{x_1, z_1}(1 - \delta_{x_2 z_2}) \\ &+ \Theta(x_3 - z_3)\delta_{x_1 z_1}\delta_{x_2 z_2} \end{aligned} \tag{1.11}$$

It is easy to see that the role of the step-like function $\Theta(x)$ could be played by one of the three Heaviside functions: $\Theta_s(x)$ - the symmetric unit-valued function, $\Theta^{\pm}(x)$ - the asymmetric unit-valued functions. One should deal with $\Theta_s(x)$ with a special care.

We will not explore here various topological aspects of the $w(\mathbf{x}, \mathbf{z})$, that were briefly discussed in the papers [22–24] for the 2D case, and in the paper [25] for the 3D case, although from a different point of view. More specifically, in the paper [24] the generalized JW transformations for the 3D case are interpreted as gauge transformations with the topological charge equal to 1. These transformations are more complicated than the transformations (1.11) from the paper [25].

Here we will consider generalized transformations of the JW type for lattice systems from a different point of view. We will show that solutions (1.10) and (1.11) of the equation (1.9) are not unique. More precisely, we will show that for the 2D case, as well as for the 3D case (and also in higher dimensions), there is a possibility to introduce two or more sets of Fermi creation and annihilation operators. Moreover, there is a nontrivial transposition algebra between various sets of Fermi operators. This fact was not realized by the authors of the papers [22–25], because of, as it seems to us, not a very clear and simple enough notation. Below we will point out some possible applications of the generalized transformations of the JW type in the form postulated by us to the analysis of lattice models of statistical physics and in the graph theory, connected with the problem of calculation of generating functions for self-avoiding walks (Hamiltonian cycles on a simple rectangular lattice , see [19,27,28]).

This problem is still under investigation, what could be seen e.g. from the recent paper of Gujrati [26] devoted to a geometric description of phase transitions in terms of diagrams and their growth functions. Moreover, as far as the author's knowledge is concerned, the multidimensional transformations of the JW type were not given in such a simple and convenient form, and their properties were not examined in the sense discussed above (algebra of transposition relations etc.). In what follows we adopt the notation accepted in contemporary literature [2,4,12,14,15] and we keep in touch with the spirit and ideas of the pioneering paper of Jordan and Wigner [1].

II. THE TWO-DIMENSIONAL TRANSFORMATION OF JORDAN-WIGNER TYPE

Let us introduce three sets of $2^{\mathfrak{nm}}$ dimensional Pauli matrices $\tau_{nm}^{x,y,z}$ which can be defined in the following way [3], $\{n(m) = 1, 2, \ldots = \mathfrak{n}(\mathfrak{m})\}$:

$$\begin{aligned}
\tau_{nm}^{x} &= \mathbb{1} \bigotimes \mathbb{1} \bigotimes \mathbb{1} \ldots \mathbb{1} \bigotimes \tau^{x} \bigotimes \mathbb{1} \ldots \mathbb{1} \bigotimes \mathbb{1} \bigotimes \mathbb{1} \qquad && (\mathfrak{nm} - \text{factors}) \\
\tau_{nm}^{y} &= \mathbb{1} \bigotimes \mathbb{1} \bigotimes \mathbb{1} \ldots \mathbb{1} \bigotimes \tau^{y} \bigotimes \mathbb{1} \ldots \mathbb{1} \bigotimes \mathbb{1} \bigotimes \mathbb{1} \qquad && (\mathfrak{nm} - \text{factors}) \\
\tau_{nm}^{z} &= \mathbb{1} \bigotimes \mathbb{1} \bigotimes \mathbb{1} \ldots \mathbb{1} \bigotimes \tau^{z} \bigotimes \mathbb{1} \ldots \mathbb{1} \bigotimes \mathbb{1} \bigotimes \mathbb{1} \qquad && (\mathfrak{nm} - \text{factors})
\end{aligned} \tag{2.1}$$

where the standard Pauli matrices $\tau^{x,y,z}$ are situated in these tensor products at the nm-th place ($\mathbb{1}$ - the unit 2×2 matrix). Further, the doubly indexed Pauli operators $\tau_{nm}^{\pm}$ are defined in the way analogous, i.e. as

$$\tau_{nm}^{\pm} = \frac{1}{2}\left(\tau_{nm}^{z} \pm i\tau_{nm}^{y}\right), \quad \tau_{nm}^{x} = -2\left(\tau_{nm}^{+}\tau_{nm}^{-} - \frac{1}{2}\right), \quad \tau_{nm}^{z} = \tau_{nm}^{+} + \tau_{nm}^{-} \tag{2.2}$$

The Pauli operators satisfy anticommutation transposition relations for the same lattice node (nm):

$$\left\{\tau_{nm}^{+}, \tau_{nm}^{-}\right\}_{+} = 1, \quad \left(\tau_{nm}^{+}\right)^{2} = \left(\tau_{nm}^{-}\right)^{2} = 0, \tag{2.3}$$

and commutation relations for different nodes:

$$\left[\tau^{\pm}_{nm}, \tau^{\pm}_{n'm'}\right]_{-} = 0 \qquad\qquad (nm) \neq (n'm'). \tag{2.4}$$

In other words the Pauli operators (2.2-2.4) behave as Fermi operators for the same node and as Bose operators for different nodes. To accomplish the transition to the Fermi representation for the whole lattice i.e. to the representation of Fermi creation and annihilation operators $(c^{\dagger}_{nm}, c_{nm})$ for the whole lattice, we introduce, in analogy to the one-dimensional J-W transformations (1.1), (1.4) two- dimensional transformations of J-W type, enabling us to express the Fermi operators for a two-dimensional system $(c^{\dagger}_{nm}, c_{nm})$ by Pauli operators $(\tau^{\pm}_{nm})$. It occurs that in the two-dimensional case there exist two sets of such transformations, (we do not include here the inverse transformations of the type (1.5)), which we write in the form:

$$\begin{aligned}
\alpha^{\dagger}_{nm} &= \exp\left(i\pi\sum_{k=1}^{n-1}\sum_{l=1}^{\mathfrak{m}}\tau^{+}_{kl}\tau^{-}_{kl} + i\pi\sum_{l=1}^{m-1}\tau^{+}_{nl}\tau^{-}_{nl}\right)\tau^{+}_{nm},\\
\alpha_{nm} &= \exp\left(i\pi\sum_{k=1}^{n-1}\sum_{l=1}^{\mathfrak{m}}\tau^{+}_{kl}\tau^{-}_{kl} + i\pi\sum_{l=1}^{m-1}\tau^{+}_{nl}\tau^{-}_{nl}\right)\tau^{-}_{nm},
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
\beta^{\dagger}_{nm} &= \exp\left(i\pi\sum_{k=1}^{\mathfrak{n}}\sum_{l=1}^{m-1}\tau^{+}_{kl}\tau^{-}_{kl} + i\pi\sum_{k=1}^{n-1}\tau^{+}_{km}\tau^{-}_{km}\right)\tau^{+}_{nm},\\
\beta_{nm} &= \exp\left(i\pi\sum_{k=1}^{\mathfrak{n}}\sum_{l=1}^{m-1}\tau^{+}_{kl}\tau^{-}_{kl} + i\pi\sum_{k=1}^{n-1}\tau^{+}_{km}\tau^{-}_{km}\right)\tau^{-}_{nm}.
\end{aligned} \tag{2.6}$$

It is straightforward to check, using the relations (2.3) and (2.4) that the operators $(\alpha^{\dagger}_{nm}, \alpha_{nm})$ and $(\beta^{\dagger}_{nm}, \beta_{nm})$ are actually Fermi operators for all the lattice i.e. that they satisfy anticommutation transposition relations for all nodes:

$$\begin{aligned}
\{\alpha^{\dagger}_{nm}, \alpha_{nm}\}_{+} = 1, \quad (\alpha^{\dagger}_{nm})^2 = (\alpha_{nm})^2 = 0,\\
\{\alpha^{\dagger}_{nm}, \alpha^{\dagger}_{n'm'}\}_{+} = \{\alpha^{\dagger}_{nm}, \alpha_{n'm'}\}_{+} = \{\alpha_{nm}, \alpha_{n'm'}\}_{+} = 0 \qquad (nm) \neq (n'm'),
\end{aligned} \tag{2.7}$$

and analogously for β-operators. The inverse transformations to (2.5-2.6) are:

$$\begin{aligned}
\tau^{+}_{nm} &= \exp\left(i\pi\sum_{k=1}^{n-1}\sum_{l=1}^{\mathfrak{m}}\alpha^{\dagger}_{kl}\alpha_{kl} + i\pi\sum_{l=1}^{m-1}\alpha^{\dagger}_{nl}\alpha_{nl}\right)\alpha^{\dagger}_{nm},\\
\tau^{+}_{nm} &= \exp\left(i\pi\sum_{k=1}^{\mathfrak{n}}\sum_{l=1}^{m-1}\beta^{\dagger}_{kl}\beta_{kl} + i\pi\sum_{k=1}^{n-1}\beta^{\dagger}_{km}\beta_{km}\right)\beta^{\dagger}_{nm},
\end{aligned} \tag{2.8}$$

and analogously for τ^{-}_{nm}.

The checking procedure could be easily performed provided we take into consideration the following relations:

$$\exp\left(i\pi\sum_{nm}\tau^+_{nm}\tau^-_{nm}\right)=\prod_{nm}(1-2\tau^+_{nm}\tau_{nm})=\prod_{nm}(\tau^x_{nm}) \tag{2.9}$$

That's obvious:

$$\tau^+_{nm}\tau^-_{nm}=\alpha^{dag}_{nm}\alpha_{nm},\quad \tau^+_{nm}\tau^-_{nm}=\beta^\dagger_{nm}\beta_{nm},\quad \alpha^\dagger_{nm}\alpha_{nm}=\beta^\dagger_{nm}\beta_{nm} \tag{2.10}$$

The last formula in (2.10) expresses the condition of local equality of occupation numbers for α- and β-fermions in the same node.

It is a simple matter to see that in the discrete case the solution (1.10) can be identified with the first pair of transformations (2.8), if we introduce the following correspondence:

$$x_1\to n,\qquad z_1\to k;\qquad x_2\to m,\qquad z_2\to l \tag{2.11}$$

Afterwards, we have

$$\sum_{\mathbf{z}} w(\mathbf{x},\mathbf{z})\alpha^\dagger(\mathbf{z})\alpha(\mathbf{z})\to\sum_{k=1}^{n-1}\sum_{l=1}^{M}\alpha^\dagger_{kl}\alpha_{kl}+\sum_{l=1}^{m-1}\alpha^\dagger_{nl}\alpha_{nl}$$

Analogously, for

$$x_1\to m,\qquad z_1\to l;\qquad x_2\to n,\qquad z_2\to k, \tag{2.12}$$

we obtain the second pair of transformations (2.8). From (2.11-2.12) it follows that the pair of transformations (2.8) can be written in the form (in the notation from [25]):

$$w(\mathbf{x},\mathbf{z})=\Theta(x_i-z_i)(1-\delta_{x_iz_i})+\Theta(x_j-z_j)\delta_{x_iz_i},\qquad (i\neq j)=1,2. \tag{2.13}$$

From (2.13) drop out two others inverse transformations, analogous to transformations (1.5) in the one-dimensional case. Obviously, (2.11-2.12) corresponded to the symmetric transposition group S_2 and, therefore, to the complete set of transformations for the 2D case in the discrete case corresponds the group S_2. This set of transformations should be complemented by a group of inverse transformations, analogous to transformations (1.5) in one-dimensional case. These inverse transformations could always be written out if necessary.

It follows from (2.5–2.6), (2.8) and (2.10) that operators $(\alpha^\dagger_{nm}, \alpha_{nm})$ and $(\beta^\dagger_{nm}, \beta_{nm})$ are connected by relations of the form:

$$\alpha^\dagger_{nm} = \exp(i\pi\phi_{nm})\beta^\dagger_{nm}, \qquad \alpha_{nm} = \exp(i\pi\phi_{nm})\beta_{nm}$$
$$\phi_{nm} = \left[\sum_{k=n+1}^{\mathfrak{n}} \sum_{l=1}^{m-1} + \sum_{k=1}^{n-1} \sum_{l=m+1}^{\mathfrak{m}}\right] \alpha^\dagger_{kl}\alpha_{kl} = [\ldots]\, \beta^\dagger_{kl}\beta_{kl} \tag{2.14}$$

It is obvious that the operators ϕ_{nm} commute with operators $(\alpha^\dagger_{nm}, \alpha_{nm})$ and $(\beta^\dagger_{n'm'}, \beta_{n'm'})$ in the same node:

$$[\phi_{nm}, \alpha^\dagger_{nm}]_- = \ldots = \ldots = [\phi_{nm}, \beta_{nm}]_- = 0, \tag{2.15}$$

because the occupation numbers with the index (nm) drop out of the ϕ_{nm}.

It is easy to see that in the one-dimensional case the transformations introduced become identical to the one-dimensional J-W transformations (1.1), (1.4). We should stress here that an inverse transition does not exist, i.e. a transformation from the one-dimensional J-W transitions (1.1), (1.4) to their two-dimensional analogue (2.5- 2.6). In other words a "derivation" of the two-dimensional transformations (2.5-2.6) from the one-dimensional J-W transformations (1.1), (1.4) is not possible using, for example, the lexicological order [21] for the current double indexed variables $\tau^\pm_{nm}$, or using different types of order. No doubts, we obtain in this case the well known one- dimensional transformations, not more. Of course, we use extensively the ideas of Jordan and Wigner to obtain the multidimensional analogue of the transformations.

Now, let us find the transposition relations for the operators $(\alpha^\dagger_{nm}, \alpha_{nm})$ and $(\beta^\dagger_{n'm'}, \beta_{n'm'})$. Firstly, using the relations:

$$\exp(i\pi\alpha^\dagger_{nm}\alpha_{nm}) = (1 - 2\alpha^\dagger_{nm}\alpha_{nm}) = (-1)^{\alpha^\dagger_{nm}\alpha_{nm}}, \tag{2.16}$$

it is easy to find the following transposition relations:

$$\{(-1)^{\alpha^\dagger_{nm}\alpha_{nm}}, \alpha_{nm}\}_+ = \{(-1)^{\alpha^\dagger_{nm}\alpha_{nm}}, \alpha^\dagger_{nm}\}_+ = \{(-1)^{\alpha^\dagger_{nm}\alpha_{nm}}, \beta_{nm}\}_+ = \{(-1)^{\alpha^\dagger_{nm}\alpha_{nm}}, \beta^\dagger_{nm}\}_+ = 0 \tag{2.17}$$

where the equality of occupation numbers for the α and β fermions (2.10) has been used. The straightforward calculation gives the following transposition relations:

$$\{\alpha^{\dagger}_{nm}, \beta_{nm}\}_{+} = \{\beta^{\dagger}_{nm}, \alpha_{nm}\}_{+} = (-1)^{\phi_{nm}} \tag{2.18}$$

$$[\alpha_{nm}, \beta_{n'm'}]_{-} = \ldots = \left[\alpha^{\dagger}_{nm}, \beta^{\dagger}_{n'm'}\right]_{-} = 0, \text{for} \begin{pmatrix} n' \leq n-1, \;\; m' \geq m+1 \\ n' \geq n+1, \;\; m' \leq m-1 \end{pmatrix} \tag{2.19}$$

$$\{\alpha_{nm}, \beta_{n'm'}\}_{+} = \{\alpha^{\dagger}_{nm}, \beta^{\dagger}_{n'm'}\}_{+} = 0 \tag{2.20}$$

for all the cases, where the operators ϕ_{nm} in (2.18) are defined by the formula (2.14) and we use the equality:

$$\exp(i\pi\phi_{nm}) = (-1)^{\phi_{nm}}.$$

The transformation relations (2.18-2.20) are simply illustrated in the Fig.1, where the distinguished operator α_{nm} for a fixed node (nm) commutes with β-operators for the nodes (n', m'), denoted by a cross $(\times)$. For all other nodes α- and β-operators anticommute. On the other hand, we can easily obtain the expression for the commutation for the same node (nm)

$$\alpha_{nm}\beta^{\dagger}_{nm} - \beta^{\dagger}_{nm}\alpha_{nm} = (-1)^{\beta^{\dagger}_{nm}\beta_{nm}}(-1)^{\phi_{nm}} \tag{2.21}$$

In this way we obtain a rather specific structure of transposition relations for α- and β-operators for the lattice, in spite of the fact that this structure has some symmetry.

Here is the proper place to make a small digression and compare the situation described above with the situation in which we use the method of second quantization. In the latter case for a system constituting of different particles the operators of second quantization are introduced. The operators that are assigned to bosons and fermions commute. On the other hand, for the operators assigned to different fermions it is usually postulated without proof [14] that in the framework of the non-relativistic theory these operators could be formally treated either as commuting or as anticommuting. For both possible assumptions about the transposition relations the application of the method of second quantization gives

the same result. On the other hand, as far as the relativistic theory is concerned, where some transmutations of particles are possible, we should treat the creation and annihilation operators for different fermions as anticommuting.

In the case we deal with, we operate formally with "quasiparticles" of the α- and β-type, which, treated separately, are subjected to the Fermi statistics. In contrast, the transposition relations among the members of these two sets of operators depend on the relative position of the "quasiparticles" in the nodes of the lattice. As far as it is known to the author, such a situation has not occured in quantum physics.

The fact that in the two-dimensional case there are two sets of transformations (neglecting the inversional transformations of the type (1.5), discussed below) of the JW type (2.5) and (2.6) is, in a way, justified if we consider statistical mechanics of two- and three- dimensional lattice models with the nearest-neighbours interactions. Namely, assume we managed for one of such models (for example, for the Ising model or another model describing two level states of any system) to express the Hamiltonian in terms of the doubly indexed Pauli operators $\tau^{\pm}_{nm}$ and the components of the Hamiltonian are of the form:

$$\tau^{+}_{nm}\tau^{+}_{n+1,m}, \qquad \tau^{+}_{nm}\tau^{-}_{n,m+1}, \quad etc. \tag{2.22}$$

Then it is easy to obtain for the first component of (2.22), after application of the transformations (2.6), the expression:

$$\tau^{+}_{nm}\tau^{+}_{n+1,m} = \beta^{\dagger}_{nm}(1-2\beta^{\dagger}_{nm}\beta_{nm})\beta^{\dagger}_{n+1,m} = \beta^{\dagger}_{nm}\beta^{\dagger}_{n+1,m}, \tag{2.23}$$

and for the second component of (2.22), after application of transformations (2.5), the expression:

$$\tau^{+}_{nm}\tau^{-}_{n,m+1} = \alpha^{\dagger}_{nm}(1-2\alpha^{\dagger}_{nm}\alpha_{nm})\alpha_{n,m+1} = \alpha^{\dagger}_{nm}\alpha_{n,m+1}, \tag{2.24}$$

On the other hand, if we apply to the first component of (2.22) the transformations (2.5), we obtain:

$$\tau^+_{nm}\tau^+_{n+1,m} = (-1)^{\phi_{nm}}\alpha^\dagger_{nm}(-1)^{\phi_{n+1,m}}\alpha^\dagger_{n+1,m} = (-1)^{\chi_{nm}}\alpha^\dagger_{nm}\alpha^\dagger_{n+1,m},$$
$$\chi_{nm} = \sum_{l=m+1}^{\mathfrak{m}} \alpha^\dagger_{nl}\alpha_{nl} + \sum_{l=1}^{m-1} \alpha^\dagger_{n+1,l}\alpha_{n+1,l} \tag{2.25}$$

and if we apply to the second component of (2.22) the transformations (2.6), we obtain:

$$\tau^+_{nm}\tau^-_{n,m+1} = (-1)^{\phi_{nm}}\beta^\dagger_{nm}(-1)^{\phi_{n,m+1}}\beta_{n,m+1} = (-1)^{\rho_{nm}}\beta^\dagger_{nm}\beta_{n,m+1},$$
$$\rho_{nm} = \sum_{k=n+1}^{\mathfrak{n}} \beta^\dagger_{km}\beta_{km} + \sum_{k=1}^{n-1} \beta^\dagger_{k,m+1}\beta_{k,m+1} \tag{2.26}$$

In this way, in the expressions (2.25) and (2.26) some unpleasant phase factors $(-1)^{\chi_{nm}}$ and $(-1)^{\rho_{nm}}$ are contained. It is this place where certain difficulties appear, connected with the attempts to apply the JW transformations (2.8) in the solution of the 3D Ising model using the approach introduced in the paper [12]. In some cases, for example during the calculation of energy of the ground state, those phase factors could be eliminated from considerations, because:

$$(-1)^{\chi_{nm}}|0\rangle = 1|0\rangle, \qquad (-1)^{\rho_{nm}}|0\rangle = 1|0\rangle \tag{2.27}$$

where: $|0\rangle$ - the vacuum state; $\alpha_{nm}|0\rangle = \beta_{nm}|0\rangle = 0$. As we see, application of the transformations (2.6) to the first index (n) and the transformations (2.5) to the second index (m) does not lead to occurrence of these phase factors. As a result, this fact implies the possibility of diagonalization by transformation to the momentum representation. Unfortunately, there occur some other obstacles for the diagonalization, which will be discussed elsewhere.

The arguments given above for existence of at least two sets of transformations (2.5) and (2.6) of the J-W type in the two-dimensional case are, of course not, rigorous and are presented here only as some guiding devices. Possible future physical interpretation of these results has nothing to do with the mathematical fact of existence of two nontrivial transformations for two-dimensional systems. Introduction of inverse transformations of the type (1.5) for the two-dimensional case do not lead to any new transformations, and do not change the symmetry of the transposition relations. As it is seen below, in the three-dimensional case the situation is much more complicated and the arguments given above are, in general, powerless.

In the papers [18,19,27,28] of the author was shown a nontrivial example of application of a pair of the JW transformations (2.8) to the problem of deriving the generating function for Hamiltonian cycles on a simple rectangular lattice with $N \times M$ nodes. One of the key moments in the papers is simultaneous application of the pair of JW transformations (2.8), and not of only of one of them. We will investigate in detail application of the pair (2.8) to one of the possible solutions of the 2D Ising-Onsager problem in the external field [19].

III. THE THREE-DIMENSIONAL TRANSFORMATION OF THE J-W TYPE

In the three-dimensional case we introduce three sets of $2^{\mathfrak{nmk}}$ -dimensional Pauli matrices $\tau^{x,y,z}_{nmk}$ ($n = 1, 2, \ldots, \mathfrak{n}$; $m = 1, 2, \ldots, \mathfrak{m}$; $k = 1, 2, \ldots, \mathfrak{k}$), which are defined analogously to the two-dimensional case (2.1). Further we introduce three-index Pauli operators $\tau^{\pm}_{nmk}$ by formulae:

$$\tau^{\pm}_{nmk} = 2^{-1}(\tau^{z}_{nmk} \pm i\tau^{y}_{nmk}) \tag{3.1}$$

which satisfy anticommutation transposition relations for the same lattice node (nmk):

$$\left\{\tau^{+}_{nmk}, \tau^{-}_{nmk}\right\}_{+} = 1, \quad \left(\tau^{+}_{nmk}\right)^2 = \left(\tau^{-}_{nmk}\right)^2 = 0 \tag{3.2}$$

and commutation relations for different lattice nodes:

$$\left[\tau^{\pm}_{nmk}, \tau^{\pm}_{n'm'k'}\right]_{-} = 0 \qquad\qquad (nmk) \neq (n'm'k'). \tag{3.3}$$

It occurs that in the three-dimensional case there exist six (not including the inverse transformations of the type 1.5) sets of transformations of the J-W type, which could be represented in the form:

$$\begin{aligned}
\tau^{+}_{nmk} &= \exp\left[i\pi\left(\sum_{s=1}^{\mathfrak{n}}\sum_{p=1}^{\mathfrak{m}}\sum_{q=1}^{k-1}\alpha^{\dagger}_{spq}\alpha_{spq} + \sum_{s=1}^{\mathfrak{n}}\sum_{p=1}^{m-1}\alpha^{\dagger}_{spk}\alpha_{spk} + \sum_{s=1}^{n-1}\alpha^{\dagger}_{smk}\alpha_{smk}\right)\right]\alpha^{\dagger}_{nmk} \\
\tau^{+}_{nmk} &= \exp\left[i\pi\left(\sum_{s=1}^{\mathfrak{n}}\sum_{p=1}^{\mathfrak{m}}\sum_{q=1}^{k-1}\beta^{\dagger}_{spq}\beta_{spq} + \sum_{s=1}^{n-1}\sum_{p=1}^{\mathfrak{m}}\beta^{\dagger}_{spk}\beta_{spk} + \sum_{p=1}^{m-1}\beta^{\dagger}_{npk}\beta_{npk}\right)\right]\beta^{\dagger}_{nmk}
\end{aligned} \tag{3.4}$$

(3.5)

$$\tau^+_{nmk} = \exp\left[i\pi\left(\sum_{s=1}^{\mathfrak{n}}\sum_{p=1}^{m-1}\sum_{q=1}^{\mathfrak{k}}\gamma^\dagger_{spq}\gamma_{spq} + \sum_{s=1}^{\mathfrak{n}}\sum_{q=1}^{k-1}\gamma^\dagger_{smq}\gamma_{smq} + \sum_{s=1}^{n-1}\gamma^\dagger_{smk}\gamma_{smk}\right)\right]\gamma^\dagger_{nmk}$$

(3.6)

$$\tau^+_{nmk} = \exp\left[i\pi\left(\sum_{s=1}^{\mathfrak{n}}\sum_{p=1}^{m-1}\sum_{q=1}^{\mathfrak{k}}\eta^\dagger_{spq}\eta_{spq} + \sum_{s=1}^{n-1}\sum_{q=1}^{\mathfrak{k}}\eta^\dagger_{smq}\eta_{smq} + \sum_{q=1}^{k-1}\eta^\dagger_{nmq}\eta_{nmq}\right)\right]\eta^\dagger_{nmk}$$

(3.7)

$$\tau^+_{nmk} = \exp\left[i\pi\left(\sum_{s=1}^{n-1}\sum_{p=1}^{\mathfrak{m}}\sum_{q=1}^{\mathfrak{k}}\omega^\dagger_{spq}\omega_{spq} + \sum_{p=1}^{\mathfrak{m}}\sum_{q=1}^{k-1}\omega^\dagger_{npq}\omega_{npq} + \sum_{p=1}^{m-1}\omega^\dagger_{npk}\omega_{npk}\right)\right]\omega^\dagger_{nmk}$$

(3.8)

$$\tau^+_{nmk} = \exp\left[i\pi\left(\sum_{s=1}^{n-1}\sum_{p=1}^{\mathfrak{m}}\sum_{q=1}^{\mathfrak{k}}\theta^\dagger_{spq}\theta_{spq} + \sum_{p=1}^{m-1}\sum_{q=1}^{\mathfrak{k}}\theta^\dagger_{npq}\theta_{npq} + \sum_{q=1}^{k-1}\theta^\dagger_{nmq}\theta_{nmq}\right)\right]\theta^\dagger_{nmk}$$

(3.9)

and analogously for the operators τ^-_{nmk}. For the sake of completeness of the exposition, we have written here six transformations, which enable us to express the Pauli operators $\tau^\pm_{nmk}$ by the Fermi creation and annihilation operators $(\alpha^\dagger_{nmk}, \alpha_{nmk}, \ldots, \theta_{nmk})$. Applying the formulae (3.2), (3.3) and the formulae of the type (2.17), it is easy to show that the operators $(\alpha^\dagger_{nmk}, \alpha_{nmk}, \ldots, \theta_{nmk})$ satisfy the anticommutation transposition relations:

$$\{\alpha^\dagger_{nmk}, \alpha_{nmk}\}_+ = 1, \quad (\alpha^\dagger_{nmk})^2 = (\alpha_{nmk})^2 = 0$$
$$\{\alpha^\dagger_{nmk}, \alpha^\dagger_{n'm'k'}\}_+ = \ldots = \{\alpha_{nmk}, \alpha_{n'm'k'}\}_+ = 0 \quad (nmk) \neq (n'm'k') \tag{3.10}$$

etc., as could be straightforwardly checked. There exist also inverse transformations:

$$\alpha^\dagger_{nmk} = \exp\left[i\pi\left(\sum_{s=1}^{\mathfrak{n}}\sum_{p=1}^{\mathfrak{m}}\sum_{q=1}^{k-1}\tau^+_{spq}\tau^-_{spq} + \sum_{s=1}^{\mathfrak{n}}\sum_{p=1}^{m-1}\tau^+_{spk}\tau^-_{spk} + \sum_{s=1}^{n-1}\tau^+_{smk}\tau^-_{smk}\right)\right]\tau^+_{nmk} \tag{3.11}$$

etc., from which we can obtain easily the equations:

$$\tau^+_{nmk}\tau^-_{nmk} = \alpha^\dagger_{nmk}\alpha_{nmk} = \beta^\dagger_{nmk}\beta_{nmk} = \gamma^\dagger_{nmk}\gamma_{nmk} = \eta^\dagger_{nmk}\eta_{nmk} =$$
$$= \omega^\dagger_{nmk}\omega_{nmk} = \theta^\dagger_{nmk}\theta_{nmk} \tag{3.12}$$

using relations of the type (2.9), written for the three-dimensional case. The relations (3.12) express the conditions of equality of local occupation numbers for α- ,β- , γ- , η- , ω- and

θ- fermions for the same lattice node (nmk). In analogy to the two- dimensional case, the operators $(\alpha^\dagger_{nmk}, \ldots, \theta_{nmk})$ are connected by canonical nonlinear transformations:

$$\begin{aligned}
\alpha^\dagger_{nmk} &= (-1)^{\phi_{nmk}} \beta^\dagger_{nmk} \\
\alpha_{nmk} &= (-1)^{\phi_{nmk}} \beta_{nmk} \\
\phi_{nmk} &= \left[\sum_{s=n+1}^{\mathfrak{n}} \sum_{p=1}^{m-1} + \sum_{s=1}^{n-1} \sum_{p=m+1}^{\mathfrak{m}} \right] \alpha^\dagger_{spk} \alpha_{spk}; \qquad (3.13) \\
\alpha^\dagger_{nmk} &= (-1)^{\psi_{nmk}} \gamma^\dagger_{nmk} \\
\alpha_{nmk} &= (-1)^{\psi_{nmk}} \gamma_{nmk} \\
\psi_{nmk} &= \sum_{s=1}^{\mathfrak{n}} \left[\sum_{p=m+1}^{\mathfrak{m}} \sum_{q=1}^{k-1} + \sum_{p=1}^{m-1} \sum_{q=k+1}^{\mathfrak{k}} \right] \alpha^\dagger_{spq} \alpha_{spq}; \qquad (3.14) \\
\beta^\dagger_{nmk} &= (-1)^{\chi_{nmk}} \gamma^\dagger_{nmk} \\
\beta_{nmk} &= (-1)^{\chi_{nmk}} \gamma_{nmk} \\
\chi_{nmk} &= \sum_{s=1}^{\mathfrak{n}} \left[\sum_{p=m+1}^{\mathfrak{m}} \sum_{q=1}^{k-1} + \sum_{p=1}^{m-1} \sum_{q=k}^{\mathfrak{k}} \right] \beta^\dagger_{spq} \beta_{spq} + \sum_{s=1}^{n-1} \sum_{p=1}^{\mathfrak{m}} \beta^\dagger_{spk} \beta_{spk} \\
&\quad + \sum_{p=1}^{m-1} \beta^\dagger_{npk} \beta_{npk} + \sum_{s=1}^{n-1} \beta^\dagger_{smk} \beta_{smk}; \qquad (3.15)
\end{aligned}$$

and 12 pairs of further transformations, which could be easily written, if necessary. The operators ϕ_{nmk}, ψ_{nmk} etc. obviously commute with the operators $(\alpha^\dagger_{nmk}, \ldots, \theta_{nmk})$ in the same node, because of lack in the operators ϕ_{nmk}, ψ_{nmk} etc. of the operators of occupation numbers indexed by (nmk). It is also a rather easy task to prove that the transformations of the J-W type introduced above by the formulae (3.4-3.9) for the three-dimensional case reduce to transformations (2.5), (2.6) in the two-dimensional case.

Similarly as in the 2D case, correspondence between transformations (3.4-3.9) and the solution (12) can be established in the three-dimensional case on the basis of the symmetric group S_3. Indeed, the transformations (3.4-3.9) in the notation from the paper [25] can be written in the form:

$$\begin{aligned}
w(\mathbf{x}, \mathbf{z}) &= \Theta(x_i - z_i)(1 - \delta_{x_i z_i}) + \Theta(x_j - z_j)\delta_{x_i z_i}(1 - \delta_{x_j z_j}) \\
&\quad + \Theta(x_k - z_k)\delta_{x_i z_i}\delta_{x_j z_j}, \\
(i \neq j \neq k) &= 1, 2, 3. \qquad (3.16)
\end{aligned}$$

It should be noticed that from (3.15) drop out the inverse transformations of the type (1.5), generalized to the 3D case. Obviously in the d-dimensional case the complete number of transformations of the JW type is equal to d! (neglecting the inverse transformations), and the correspondence can be established using the symmetric group S_d.

There are no principal obstacles against full analysis of transposition relations for the operators $(\alpha^{\dagger}_{nmk}, \ldots, \theta_{nmk})$ and, if necessary, we can write all the relations we want. Here we consider the transposition relations only for the operators $(\alpha^{\dagger}_{nmk}, \beta^{\dagger}_{n'm'k'}, \gamma^{+}_{n''m''k''},)$, to get feeling of the "geometric structure" of these relations for the three- dimensional case. First of all let us observe that, according to (3.13), transposition relations for $(\alpha^{\dagger}_{nmk}, \alpha_{nmk})$ and $(\beta^{\dagger}_{n'm'k'}, \beta_{n'm'k'})$ for $k = k'$ are of the form (2.18-2.20), where it is sufficient to add the third index k to all the operators. In other words, transposition relations in the plane $(nm/k = const)$ for α- and β- operators behave like in the two- dimensional case. This fact could be expected, because for the fixed k we deal actually with two-dimensional transformations of the J-W type (2.5 -2.6). It is easy to see that for all of the three mutually orthogonal planes there exists a pair of operators, for which the transformation relations are of the form (2.18-2.20). In accordance with (3.4- 3.9) for the plane $(mk/n = const)$ such a pair will be the pair of operators $\omega - \theta$, and for the plane $(nk/m = const)$ - the pair of operators $\gamma - \eta$. Further, according to (3.13) for $(k \neq k')$, α- and β - operators anticommute for any (nm) i.e.

$$\{\alpha^{\dagger}_{nmk}, \beta^{\dagger}_{n'm'k'}\}_{+} = \ldots = \{\alpha_{nmk}, \beta_{n'm'k'}\}_{+} = 0, \qquad (k \neq k') \tag{3.17}$$

It is obvious that also the pairs of operators $\omega - \theta$ anticommute for $(n \neq n')$ and the pair of operators $\gamma - \eta$ anticommutes for $(m \neq m')$. Now, from (3.14) there follow transposition relations for α- and γ-operators of the form analogous to (2.18-2.20) i.e.:

$$[\alpha_{nmk}, \gamma_{n'm'k'}]_{-} = \ldots = \ldots = 0, \qquad \text{for} \begin{pmatrix} m' \leq m-1 \ , \ k' \geq k+1 \\ m' \geq m+1 \ , \ k' \leq k-1 \end{pmatrix},$$
$$\{\alpha_{nmk}, \gamma_{n'm'k'}\}_{+} = \ldots = \{\alpha^{\dagger}_{nmk}, \gamma_{n'm'k'}\}_{+} = 0, \tag{3.18}$$

in all other cases, with the only difference that the equations (3.18) are satisfied for any n

and n'. Analogously, transposition relations for other pairs of operators are considered. In the general case the symmetry characteristic for the two-dimensional case (see (2.18 –2.20) and Fig.1) disappears.

Therefore, in the case of the three-dimensional space there exist six nontrivial transformations of the J-W type (3.4-3.9) and the algebra of their transposition relations is much more complicated than the analogous algebra in the two-dimensional case. Some examples of application of the transposition relations (3.4-3.9) will be considered elsewhere, where the three-dimensional Ising model with and without external magnetic field is considered from the new point of view, as well as other models of statistical mechanics and physics etc. [4,18,19,27].

We believe it is reasonable to notice here one beautiful fact connected with generalized transformations of the JW type for the $d \geq 3$ case. Namely, for lattice models with nearest neighbours interactions (for example, for the Ising model), the statistical sum of the system can be represented in terms of three-index Pauli operators $\tau^{\pm}{}_{nmk}$, which will enter the sum as bi-linear products of the type:

$$\tau^{+}_{nmk}\tau^{+}_{n+1,mk}, \tag{3.19}$$

etc., see [27,28]. Then one can easily realize that among transformations (3.4)-(3.9) there are two, (3.4) and (3.5), which after application of which to (3.18) lead to the expressions $\alpha^{\dagger}_{nmk}\alpha^{\dagger}_{n+1,mk}$ and $\gamma^{\dagger}_{nmk}\gamma^{\dagger}_{n+1,mk}$ in which the phase factors of the type $(-1)^{\chi_{nmk}}$ (2.25) etc. are not present, and the same applies to indices m and k. In other words, there exist two equivalent (in the sense of absence of the phase factors) transformations of the JW type for each degree of freedom of Pauli variables. This is a sort of degeneracy in each index. For the 3D case the number N_d of generalized transformations of the JW type is equal to $N_d = 3! = 6$, and the "degree" of degeneracy in every index is equal to $v = (N_d/d) = 2$. Then in the general case we have $v = (N_d/d) = (d!/d)$ and for $d = 2$, $v = 1$ as we have mentioned above (2.22)-(2.26).

IV. CONCLUSIONS

We believe we have menaged to show in this paper advantages and simplicity of the generalized transformations of the JW type introduced above. Especially important is the fact that this formulation of the JW transformations enabled us to find the entire sequence of sets of the transformations, and to investigate the algebra of transposition relations for various sets of Fermi operators. Moreover, as far as the analysis of topological aspects of the transformations and consideration of their continuous counterparts are concerned, it seems that the notation from the papers [22]- [25] is more convenient.

We omitted in this paper consideration of the problem of correspondence between discrete generalized transformations of the JW type, introduced in this paper and analogous transformations given in the papers [22]- [25], which are their continuous counterparts thereof. The reason is there are still many unclear points that need some analysis in future papers. Especially interesting, both from the physical (in the framework of possible applications) and mathematical point of view would be examination of the connection between our transformations and transformations taken from the paper [24]. Such analysis is missing also in the paper [25]. This statement applies to the discrete case, but to the continuous one as well provided there exists a formal way to take the continuous limit for the lattice constant $\mathbf{a}\to 0$. For example, yet in the 2D case such a formal transition to the continuous limit could result in singularities of the section type along lines $x = const$ and $y = const$, where $\alpha(x, y)$, $\beta(x, y)$- densities in a fixed point (x,y) in the chosen coordinate system ($\mathbf{e}_1$,$\mathbf{e}_2$) (relative to the transposition relations between α- and β-operators (2.18)-(2.21). Here also some other problems appear which we will be explored in future papers.

The attention of physicists and mathematicians was and still is in the field theory and in connections there of with various models of classical and quantum statistical mechanics (see for example, [2], [4], [20]) and the literature cited in these papers). It is known [2], [4], [20] that, in some cases a deep connection between the models of quantum field theory and the models of statistical mechanics has been discovered.

We hope that in the given form the proposed above JW type transformations for the 2D and 3D could be a valuable tool in the analysis of the already known models of statistical mechanics and quantum field theory, as well as they are expected to initiate formulation of new problems in these areas of theoretical physics. With the help of generalized J-W transformations type a new approach for Lenz-Ising-Onsager problem (LIO) has been made [18,19,27–29]. For example, in the frame of this approach 2D LIO problem in the asymptotic magnetic field has been solved [19]:

$$-\beta f_2(h \to 0) \sim \ln 2 + 2\ln(\cosh h/2) + \frac{1}{2\pi^2}\int_0^\pi \int_0^\pi \ln[\cosh 2K_1^* \cosh 2K_2^* - \sinh 2K_1^* \cos q - \sinh 2K_2^* \cos p] dq dp, \tag{4.1}$$

where the parameters $(K_{1,2}, h)$ are to be renormalised in the following way ($K_{1,2} \geq 0$):

$$\sinh 2K_{1,2}^* = \beta_{1,2}[\sinh 2K_{1,2}(1 - \tanh^2(h/2)],$$
$$\cosh(2K_{1,2}^*) = \beta_{1,2}[\cosh 2K_{1,2} + \tanh^2(h/2)\sinh 2K_{1,2}],$$
$$\beta_{1,2} = [1 + 2\tanh^2(h/2)\sinh 2K_{1,2} e^{2K_{1,2}}]^{-1/2}, \quad \tanh^2 h_{1,2}^* = \tanh^2(h/2)\frac{\beta_{1,2}\exp(2K_{1,2})}{\cosh^2 K_{1,2}^*}, \tag{4.2}$$

where $\alpha(h,x) = \tanh^2(h/2)(1+\cos x)/(\sin x)$ and $K_{1,2} = \beta J_{1,2}$, $h = \beta H$, $\beta = 1/k_B T$, where T denotes temperature and k_B - the Boltzman constant.

ACKNOWLEDGEMENTS

I am grateful to Dr. H. Makaruk for her help in preparation of the final form of this paper.

This paper was supported by the KBN grant N^o **2 P03B 088 15**.

REFERENCES

[1] P.Jordan, E. Wigner: Z.Physik, **47**, (1928), 631

[2] Yu. A. Izyumov, Yu. N. Skryabin: Statistical Mechanics of Magnetically Ordered Systems, Nauka, Moscow, (in Russian), 1987.

[3] K.Huang: Statistical Mechanics, J.Wiley and Sons, Inc., New York – London, 1963.

[4] R.J.Baxter: Exactly solved models in statistical mechanics, Academic Press, London, 1982.

[5] H.A.Kramers: Quantum Mechanics, New York, 1957.

[6] S.Rodriguez: Phys.Rev., **116**, (1959), 1474

[7] T.W.Ruijgrok, S.Rodriguez: Phys.Rev., **119**, (1960), 596

[8] E.Lieb, T.Schultz, D.Mattis: Ann.Phys., **16**, (1961), 407

[9] E.Lieb, D.Mattis: Phys.Rev., **125**, (1962), 164

[10] T.Schultz: J.Math.Phys., **4**, (1963), 666

[11] Y.Nambu: Prog.Theor.Phys. Japan, **5**, (1950), 1

[12] T.D.Shultz, D.C.Mattis, E.H.Lieb: Rev.Mod.Phys., **36**, (1964), 856

[13] O.Foda et al: J.Math.Phys., **35(1)**, (1994), 13

[14] L.D.Landau, E.M. Lifschitz: Quantum Mechanics, **III**, 4-th issue, Nauka, Moscow, 1989.

[15] J.M. Ziman: Models of Disorder, Univ. Press, Cambridge, 1979.

[16] L.Onsager: Phys.Rev, **65**, (1944), 117

[17] L.Onsager, B.Kaufman: Phys.Rev, **76**, (1949), 1232

[18] M.S. Kochmański, Zh.Eksp.Teor.Fiz., **111**, (1997) 1717; [JETP **84**, (1997) 940].

[19] M.S. Kochmański, Phys.Rev.E, **56**, N4, (1997) 3831; **Cond-mat-9710203**, October 1997.

[20] M.Gaudin: La Fonction d'one de Bethe, MASSON, Paris, 1983.

[21] C.A. Hurst, H.S. Green: J. Chem. Phys., **33**, (1961), 1059

[22] E.Fradkin: Phys.Rev.Lett., **63**, (1989), 322

[23] Y.R. Wang: Phys.Rev. **b43**, (1991), 3786

[24] L.Huerta, J. Zanelli: Phys. Rev.Lett. **71**, (1993), 3622

[25] S. Wang: Phys. Rev. **E51**, (1995), 1004

[26] P.D. Gujrati: Phys. Rev. **E 51**, (1995), 957

[27] M.S. Kochmański, Acta Phys.Pol.**B 29**, N 8, (1998) 1963; **Cond-mat-9710201**, October 1997.

[28] M.S. Kochmański, Lenz-Ising-Onsager problem and Hamiltonian graphs on the lattice, J.Phys.Ukr., N 7 (1998), in press.

[29] M.S. Kochmański, **Cond-mat-9807115**, July 1998; submitted to Phys.Rev.E (1998).

V. CAPTIONS FOR ILUSTRATIONS

Fig. 1 "Geometry" of transposition relations for $\alpha-$ and $\beta-$ operators:

* — $\alpha-$ operator

$\times$ — $\beta-$ operator